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Theoretical aspects of gravity–capillary waves in non-rectangular channels

By MARK D. GROVES†

School of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK

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This article reports the results of theoretical research concerning linear waves propagating on the surface of water in a uniform horizontal channel of arbitrary cross-section. Three different versions of the problem are considered. The first is the hydrodynamic problem when surface tension is neglected. The second and third include capillary effects, necessitating the use of edge conditions at the points of contact of the free edges and the channel walls. Two sets of edge constraints are used: pinned edges, where the lines of contact are fixed, and free edges, where the surface meets locally vertical walls orthogonally. These choices are physically realistic and have certain advantages for mathematical analysis.

The hydrodynamic problems are shown to have a Hamiltonian structure in which the non-local operators inherent in the water-wave problem are explicitly exhibited. The existence, properties and applications of normal-mode solutions are discussed, and a qualitative comparison of those obtained for each problem is given. Explicit and numerical calculations of the dispersion relations for the normal modes are also carried out. A long-wave theory based upon a decomposition of the hydrodynamic problems in Fourier-transform space is developed. Finally a bifurcation theory for linear travelling waves is discussed, a potential application of which is the construction of an existence theory for periodic travelling-wave solutions of the corresponding nonlinear problems.

1. Introduction

1.1. *The hydrodynamic problems*

This article is concerned with three related linear water-wave problems. All three have the same physical domain, namely a uniform horizontal channel of arbitrary cross-section (figure 1). (In fact the results stated in this paper are valid only if D is convex and its boundary ∂D is piecewise smooth. An 'arbitrary' cross-section will therefore be assumed to have these properties.) Cartesian coordinates (x, y, z) have been introduced, with x directed along the length of the channel and y vertically upwards. In its undisturbed state the fluid occupies a domain D bounded by piecewise-smooth rigid channel walls Γ and the undisturbed free surface

$$S_0 = \{(x, 0, z) : x \in \mathbb{R}, 0 \leqslant z \leqslant b\}.$$

† Present address: Department of Mathematical Sciences, Loughborough University of Technology, Loughborough, Leicestershire, LE11 3TU, UK.

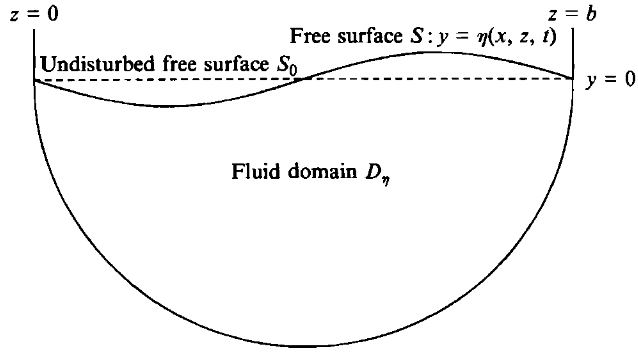

FIGURE 1. A cross-section through a uniform horizontal channel.

When in motion, the fluid domain D_η is the region bounded by Γ and the free surface S whose equation is

$$y = \eta(x,z,t), \qquad x \in \mathbb{R}, \quad 0 \leqslant z \leqslant b.$$

Two types of wave motion are studied in this paper, namely periodic waves of period ℓ and aperiodic motions that vanish as $x \to \pm\infty$. For use in the mathematical formulations below, the domains S_0 and D will henceforth be restricted to the region $0 \leqslant x \leqslant \ell$ in the case of ℓ-periodic waves.

The first hydrodynamic problem concerns linear water waves in this kind of channel in the absence of surface tension. In terms of a velocity potential $\phi(x,y,z,t)$ the mathematical problem is to solve Laplace's equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \qquad \text{in} \qquad D \tag{1.1}$$

subject to the boundary conditions

$$\frac{\partial \phi}{\partial n} = 0 \qquad \text{on} \qquad \Gamma, \tag{1.2}$$

$$\eta_t = \phi_y \qquad \text{on} \qquad S_0, \tag{1.3}$$

$$\phi_t + g\eta = 0 \qquad \text{on} \qquad S_0, \tag{1.4}$$

together with periodicity or evanescence conditions and appropriate initial data (see Lamb 1924, chapter IX; Whitham 1974, chapter 13).

The other two problems concern linear water waves in the present type of channel when surface tension is operative. To consider these problems one must replace equation (1.4) with

$$\phi_t - \sigma(\eta_{xx} + \eta_{zz}) + g\eta = 0 \qquad \text{on} \qquad S_0, \tag{1.5}$$

where $\sigma > 0$ is the coefficient of surface tension. The presence of a second derivative in this equation means that further information must be provided to complete the specification of the hydrodynamic problem. One specifies *edge constraints* at the lines $z = 0, z = b$ (e.g. see Hocking 1987). In the present paper attention is focussed on two physically realistic choices of edge constraints. One choice is to specify that the points of contact are fixed, that is

$$\eta(x,0,t) = \eta(x,b,t) = 0. \tag{1.6}$$

Physical systems with these *pinned* edges have been studied experimentally and theoretically by several authors, notably Scott & Benjamin (1978), Benjamin & Scott (1979), Benjamin (1980), Graham-Eagle (1983), Shen (1983), Benjamin & Graham-Eagle (1985) and Weidman & Norris (1987).

The second choice is to specify that the free surface touches the channel walls at a fixed angle. When capillarity is absent, the free surface meets the channel walls orthogonally. Let us use this *free-edge* condition when surface tension is included. In the channel setting outlined above, the free surface is supposed to be flat when the fluid is at rest. The free-edge model will therefore only be valid if the channel walls are locally vertical at the points of contact with the surface, in which case the edge constraints are

$$\eta_z(x,0,t) = \eta_z(x,b,t) = 0. \tag{1.7}$$

It is important to remember that edge constraints (1.7) must always be used in conjunction with the assumption that the channel walls are vertical at the points of contact with the free surface. This assumption will not be stated explicitly in the remainder of this paper, but must always be used when reference is made to edge constraints (1.7).

Other physical models have been used by researchers in the past (see Dussan V. 1979 for a general discussion of capillary effects at contact lines). In the present paper, however, attention is focussed on the pinned and free-edge conditions as specified above. Apart from their physical realism, there are significant mathematical advantages inherent in the models in that they allow the use of rigorous mathematical methods to analyse the problem. These mathematical advantages are shared by the hydrodynamic problem in which capillarity is neglected.

1.2. *Objectives of the present paper*

In this paper the main mathematical features of the three hydrodynamic problems are discussed in a descriptive mathematical manner in order to reveal certain aspects of their structure. Rigorous pure-mathematical theory has not been included. (Readers interested in such an analysis may refer to Groves 1994 for the problem in the absence of surface tension; a forthcoming article will provide the necessary theory for the capillary-wave problems.)

The discussion begins in §2 with the Hamiltonian formulation of these water-wave problems, where it is shown that they have the same Hamiltonian structure as the classical water-wave problem. The edge constraints, already discussed in §1.1, are crucial to the Hamiltonian structure. The choices of edge constraints given in §1.1 are precisely those which admit a canonical Hamiltonian structure using the Zakharov coordinates.

Section 2 also describes the mathematical operators that occur in the hydrodynamic problems. Benjamin (1984, p.47) remarked that the difficulty with the water-wave problem, even in its linear form, is that it involves non-local operators. Craig & Groves (1994) have expanded on this remark, showing that the nonlocal operator in question is an operator that maps Dirichlet data for Laplace's equation to the corresponding Neumann data. Such a Dirichlet–Neumann operator appears in the present hydrodynamic examples, and is the focus of the Hamiltonian theory in §2. When surface tension is operative, the situation becomes more complex in that the basic operator is no longer simply the Dirichlet–Neumann operator but the composition of this operator with another operator that is responsible for the higher-order derivatives in equation (1.5). An alternative Hamiltonian formulation for the capillary-wave problems is given in §2 in terms of variables that highlight the role played by this new operator.

In §3 attention is focussed on normal-mode properties of the problems. It is of course an elementary exercise to determine the dispersion relation between the speed c and wavenumber k of periodic linear travelling waves in two dimensions. The presence of three-dimensional geometry in the present hydrodynamic problems leads to the emergence of a countably infinite family of travelling-wave solutions, each with its own dispersion relation. These *normal-mode solutions* are readily obtained for a rectangular channel by an elementary separation-of-variables calculation. However for other cross-sections their existence and regularity must be proved using delicate functional-analytic arguments (see Groves 1994, §2.3). Such mathematical arguments have been successfully employed to determine the generic properties of the normal-mode solutions and the qualitative nature of their dispersion relations (see Groves 1994, §2.4). These results, together with examples, are given in §3.

Working in Fourier-transform space, one finds that any solution of the linearized hydrodynamic problem may be uniquely decomposed into a superposition of normal-mode solutions. This fact is the subject of §4, where the problem is reduced to a set of decoupled ordinary differential equations in Fourier-transform space. In terms of this decomposition one can in theory solve the physically important initial-value problem, which is discussed in detail in §4. A further useful aspect of the normal-mode composition involves the construction of long-wave approximations. A method of deriving long-wave approximations to the water-wave problem in the context of a Hamiltonian perturbation theory was explained by Craig & Groves (1994). A technique that is similar in principle is used in §5 of the present paper to derive a sequence of long-wave approximations to the channel problems.

Finally, §6 takes a brief look at a linear bifurcation theory. Regarding the speed c of uniformly travelling waves as a parameter, one may use the generic properties of the normal modes to determine when periodic wavetrains bifurcate from the zero solution and how they develop. Although this process is readily accomplished in the linear setting, it has significant implications for nonlinear existence theory, the subject of a forthcoming paper.

2. Hamiltonian structure

Let us now recall the basic definitions of Hamiltonian systems, both in finite and infinite dimensions. A *second-order canonical Hamiltonian system* is a two-component system of ordinary differential equations that takes the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial x \\ \partial H/\partial y \end{pmatrix}, \tag{2.1}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a function of the variables x and y called the *Hamiltonian*. A *second-order canonical Hamiltonian evolutionary system* is a two-component system of partial differential equations that takes the form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta \hat{H}/\delta u \\ \delta \hat{H}/\delta v \end{pmatrix}. \tag{2.2}$$

The variables u and v belong to a dense subset $\mathscr{D}$ of the Hilbert space $L^2(X)$ of functions that are square-integrable on an open subset X of $\mathbb{R}^n$. The function $\hat{H} : (L^2(X))^2 \to \mathbb{R}$ is termed the *Hamiltonian* and has the form

$$\hat{H} = \int_X H,$$

where H is the *Hamiltonian density function.* Finally, the *variational derivative of* $\hat{H}$ *with respect to* u is the unique element $\delta\hat{H}/\delta u$ of $L^2(X)$ with the property that

$$\int_X \frac{\delta \hat{H}}{\delta u} u_1 = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\hat{H}(u+\epsilon u_1, v)\bigg|_{\epsilon=0}$$

for all $u_1 \in \mathscr{D}$. The variational derivative $\delta\hat{H}/\delta v$ is defined in a similar fashion.

The Hamiltonian structure of the three hydrodynamic problems in this investigation (no capillarity, capillarity and pinned edges, capillarity and free edges meeting vertical walls) is the same as that of the classical linearized water-wave problem. That structure was discovered by Zakharov (1968) and has since been elaborated upon by many authors, most notably by Miles (1977, 1981) and Benjamin & Olver (1982). The Hamiltonian variables are η and $\Phi = \phi(x,0,z,t)$, which variables completely determine the wave motion. The shape of the free surface is determined by η, and Φ completes the specification of the mixed boundary-value problem

$$\Delta\phi = 0 \quad \text{in} \quad D, \tag{2.3}$$

$$\phi = \Phi \quad \text{on} \quad S_0, \tag{2.4}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad \Gamma, \tag{2.5}$$

together with periodicity or evanescence conditions, which problem uniquely determines ϕ in D. The total energy $\hat{H}_a$ of the wave motion (per period in the case of periodic waves) is the sum of the potential energy $\hat{V}$ and kinetic energy $\hat{K}$, which in the linearized approximation are

$$\hat{V}(\eta,\Phi) = \int_{S_0} \left\{\tfrac{1}{2}g\eta^2 + \tfrac{1}{2}\sigma(\eta_x^2+\eta_z^2)\right\} \mathrm{d}x\,\mathrm{d}z,$$

$$\hat{K}(\eta,\Phi) = \int_D \tfrac{1}{2}|\nabla\phi|^2 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{S_0} \tfrac{1}{2}\Phi\phi_y\,\mathrm{d}x\,\mathrm{d}z.$$

One may write the total energy functional in a more appealing way by making use of a *Dirichlet–Neumann operator,* defined as follows. Fix Φ, let ϕ be the unique solution of the Robin problem (2.3), (2.4), (2.5) and write

$$G_1\Phi = \phi_y|_{y=0}\,.$$

The operator G_1 is non-negative and self-adjoint on $L^2(S_0)$ (see Craig 1991), and one may use it to write $\hat{H}_a$ as

$$\hat{H}_a = \int_{S_0} \left\{\tfrac{1}{2}g\eta^2 + \tfrac{1}{2}\Phi G_1\Phi + \tfrac{1}{2}\sigma(\eta_x^2+\eta_z^2)\right\} \mathrm{d}x\,\mathrm{d}z. \tag{2.6}$$

It follows immediately from the self-adjointness of G_1 on $L^2(S_0)$ that

$$\frac{\delta \hat{H}_a}{\delta \Phi} = G_1\Phi = \phi_y|_{y=0}. \tag{2.7}$$

To calculate the variational derivative of $\hat{H}_a$ with respect to η, let $\dot{\eta}$ denote a variation in η. The corresponding first variation in $\hat{H}_a$ is

$$\dot{\hat{H}}_a = \int_{S_0} \{g\eta\dot{\eta} + \sigma\eta_x\dot{\eta}_x + \sigma\eta_z\dot{\eta}_z\}\,\mathrm{d}x\,\mathrm{d}z.$$

Integrating the second term by parts with respect to x and the third term by parts with respect to z, one finds that

$$\hat{H}_a = \int_{S_0} \{g\eta\dot{\eta} - \sigma\eta_{xx}\dot{\eta} - \sigma\eta_{zz}\dot{\eta}\}\, \mathrm{d}x\,\mathrm{d}z + [\sigma\eta_z\dot{\eta}]_{z=0}^{z=b}.$$

When σ is non-zero the term in square brackets vanishes if η satisfies (1.6) or $\dot{\eta}$ satisfies (1.7). In the latter case, η also satisfies (1.7) because $\dot{\eta}$ belongs to the same class of functions as η. In all cases it follows that

$$\frac{\delta\hat{H}_a}{\delta\eta} = g\eta - \sigma(\eta_{xx} + \eta_{zz}). \tag{2.8}$$

One finds from (2.7), (2.8) that (1.3) and (1.4) (if $\sigma = 0$) or (1.3) and (1.5) (if $\sigma \neq 0$) are equivalent to

$$\eta_t = \frac{\delta\hat{H}_a}{\delta\Phi}, \qquad \Phi_t = -\frac{\delta\hat{H}_a}{\delta\eta}, \tag{2.9}$$

which constitutes a Hamiltonian evolutionary system of the form (2.2) in terms of the variable $(\eta, \Phi)^T \in \mathscr{D} \subset (L^2(S_0))^2$.

The above formulations of the channel problems indicate that when $\sigma = 0$, the heart of the matter is the Dirichlet–Neumann operator G_1. When capillary effects are taken into account, the situation changes. To understand the effect of surface tension, let us replace the variables ϕ and Φ with

$$\chi = \left[1 - \frac{\sigma}{g}\frac{\partial^2}{\partial x^2} - \frac{\sigma}{g}\frac{\partial^2}{\partial z^2}\right]^{-1}\phi, \qquad X = \left[1 - \frac{\sigma}{g}\frac{\partial^2}{\partial x^2} - \frac{\sigma}{g}\frac{\partial^2}{\partial z^2}\right]^{-1}\Phi.$$

(The specification of this inverse operator is completed by the edge constraints (1.6) or (1.7).) Equations (1.1), (1.2), (1.3), (1.5) become

$$\chi_{xx} + \chi_{yy} + \chi_{zz} = 0 \quad \text{in} \quad D, \tag{2.10}$$

$$|\nabla\chi| \to 0 \quad \text{as} \quad x \to \pm\infty, \tag{2.11}$$

$$\frac{\partial\chi}{\partial n} = 0 \quad \text{on} \quad \Gamma, \tag{2.12}$$

$$\left[1 - \frac{\sigma}{g}\frac{\partial^2}{\partial x^2} - \frac{\sigma}{g}\frac{\partial^2}{\partial z^2}\right]\chi_y = \eta_t \quad \text{on} \quad S_0, \tag{2.13}$$

$$\chi_t + g\eta = 0 \quad \text{on} \quad S_0 \tag{2.14}$$

and the system (2.9), (2.6) transforms into the Hamiltonian evolutionary system

$$\eta_t = \frac{\delta\hat{H}_b}{\delta X}, \qquad X_t = -\frac{\delta\hat{H}_b}{\delta\eta}, \tag{2.15}$$

where

$$\hat{H}_b = \int_{S_0} \left\{\tfrac{1}{2}g\eta^2 + \tfrac{1}{2}XG_1\left[1 - \frac{\sigma}{g}\frac{\partial^2}{\partial x^2} - \frac{\sigma}{g}\frac{\partial^2}{\partial z^2}\right]X\right\}\mathrm{d}x\,\mathrm{d}z.$$

The above Hamiltonian formulation elucidates a fact that appears up until now to have been overlooked. The basic operator in the water-wave problem when surface tension is operative is not the Dirichlet–Neumann operator G_1 but rather the operator

$$F_1 = G_1\left[1 - \frac{\sigma}{g}\frac{\partial^2}{\partial x^2} - \frac{\sigma}{g}\frac{\partial^2}{\partial z^2}\right].$$

The replacement of G_l by F_l as the basic operator accounts for the extra difficulty in the water-wave problem when surface tension is taken into account.

3. Normal modes

3.1. *Gravity waves*

When $\sigma = 0$ the mathematical model is described by equations (1.1), (1.2), (1.3), (1.4). Among its classical elementary solutions are sinusoidal solutions of the form

$$\eta(x,z,t) = f(z)\mathrm{e}^{\mathrm{i}k(x-ct)}, \qquad \phi(x,y,z,t) = -\mathrm{i}kc\psi(y,z)\mathrm{e}^{\mathrm{i}k(x-ct)},$$

where c, f and ψ depend on the real number k. The functions ψ and f satisfy

$$\psi_{yy} + \psi_{zz} = k^2\psi \qquad \text{in} \qquad X, \tag{3.1}$$

$$\frac{\partial\psi}{\partial n} = 0 \qquad \text{on} \qquad \partial X \cap \Gamma, \tag{3.2}$$

$$\psi_y = f \qquad \text{on} \qquad \partial X \cap S_0 \tag{3.3}$$

with

$$k^2c^2\psi(0,z) = gf(z). \tag{3.4}$$

A solution $\{f, \psi, c^2(k)\}$ of (3.1), (3.2), (3.3), (3.4) is termed a *normal mode* with *dispersion relation* $c^2 = c^2(k)$. To establish the existence and properties of normal modes for a specified channel cross-section one requires a sophisticated mathematical treatment, details of which have been given by Groves (1994). The list below summarizes the properties of normal-mode solutions.

(*1*) *There is a countably-infinite set* $\{(f_n, \psi, c_n^2(k))\}_{n=0}^{\infty}$ *of normal modes;*
(*2*) *The sequence* $\{c_n^2\}_{n=0}^{\infty}$ *satisfies* $0 < c_0^2 \leqslant c_1^2 \leqslant c_2^2 \leqslant \cdots$ *with* $c_n^2 \to \infty$ *as* $n \to \infty$*;*
(*3*) *Each* f_n *is infinitely continuously differentiable on* $[0,b]$*;*
(*4*) *The set* $\{f_n\}_{n=0}^{\infty}$ *is complete and orthogonal in* $L^2(0,b)$*, that is*

$$\int_0^b f_n f_m \,\mathrm{d}z = 0, \qquad n \neq m, \tag{3.5}$$

and any function f *that is square-integrable on* $(0,b)$ *may be written as* $f = \sum_{n=0}^{\infty} A_n f_n$*, where the coefficients* A_m *are uniquely determined by* f*;*
(*5*) f_n *is either symmetric or antisymmetric about the line* $z = b/2$*;*
(*6*) f_n *has exactly* n *zeros in* $(0,b)$*;*
(*7*) c_n^2 *is an even function of* k*;*
(*8*) $c_n^2(k)$ *is an infinitely differentiable function of* k *for* $k \neq 0$*;*
(*9*) c_0 *has a finite value at* $k = 0$*;*
(*10*) $c_n^2 \sim O(k^{-2})$ *as* $k \to 0$ *for* $n \neq 0$*;*
(*11*) c_n^2 *is monotone decreasing for* $k > 0$.

All the properties in the above list are readily verified for a rectangular channel of depth h. A simple separation-of-variables argument shows that

$$f_n = A\cos\left(\frac{n\pi z}{b}\right), \qquad n = 0,1,2,\ldots, \tag{3.6}$$

$$\psi_n = A\frac{\cosh\left[\left(k^2 + (n\pi/b)^2\right)^{1/2}(y+h)\right]\cos\left(n\pi z/b\right)}{\left(k^2 + (n\pi/b)^2\right)^{1/2}\sinh\left[\left(k^2 + (n\pi/b)^2\right)^{1/2}h\right]}, \qquad n = 0,1,2,\ldots, \tag{3.7}$$

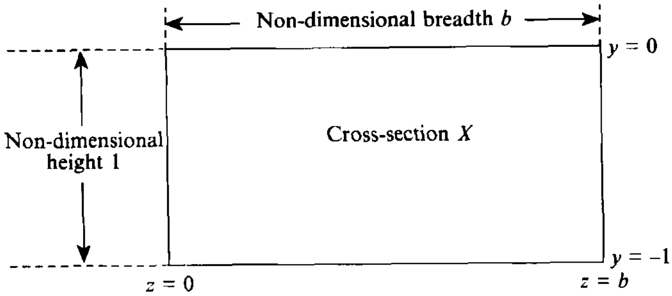

FIGURE 2. A rectangular channel.

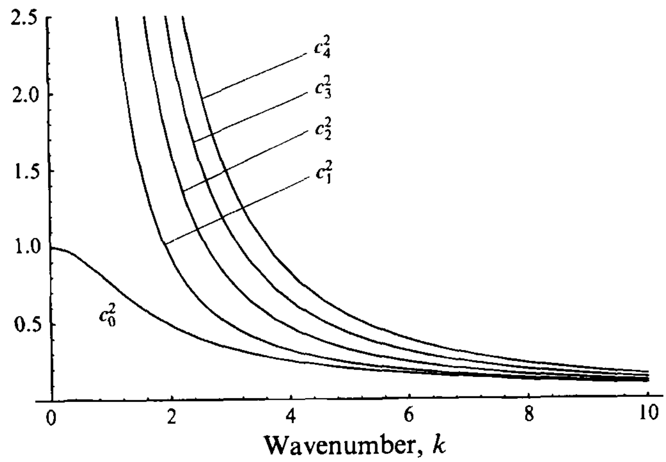

FIGURE 3. Dispersion relations for a square channel.

$$c_n^2 = \frac{g}{k^2}\left(k^2 + (n\pi/b)^2\right)^{1/2} \tanh\left[\left(k^2 + (n\pi/b)^2\right)^{1/2} h\right], \qquad n = 0, 1, 2, \ldots, \quad (3.8)$$

where A is an arbitrary constant. In order to plot the dispersion relations one introduces the dimensionless variables

$$(x, y, z) = \frac{1}{d}(x', y', z'), \qquad t = t'\left(\frac{g}{d}\right)^{1/2}, \qquad k = dk', \qquad c = c'/(gd)^{1/2}, \qquad (3.9)$$

where a prime denotes a dimensional variable and d is the mean depth of the undisturbed fluid. For a square channel the non-dimensionalization is equivalent to setting $g = h = 1$ in (3.6), (3.7), (3.8) (see figure 2). The dispersion relations for a square channel are shown in figure 3.

Remarkably, the normal modes can be explicitly calculated for some other simple channel cross-sections. In order to carry out such calculations, notice that (3.1), (3.2), (3.3), (3.4) may be combined into an eigenvalue problem for ψ, namely

$$\psi_{yy} + \psi_{zz} = k^2\psi \quad \text{in} \quad X, \qquad (3.10)$$

$$\frac{\partial\psi}{\partial n} = 0 \quad \text{on} \quad \partial X \cap \Gamma, \qquad (3.11)$$

$$\frac{g}{k^2c^2}\psi_y = \psi \quad \text{on} \quad \partial X \cap S_0. \qquad (3.12)$$

This observation provides a way of calculating the normal modes $(f_n, \psi_n, c_n^2(k))$. One

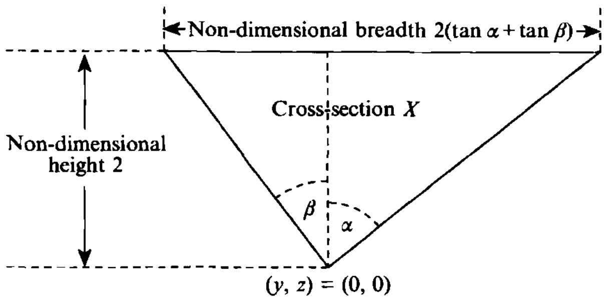

FIGURE 4. A triangular channel.

solves (3.10), (3.11), (3.12) to determine the doubles $(\psi_n, c_n^2(k))$ and the f_n are then calculated using (3.3) or (3.4).

A complete set of normal modes is known for a channel with a symmetric triangular cross-section whose walls make an angle $\pi/4$ with the vertical (see Macdonald 1894; Lamb 1924, §261; Groves 1994). In a coordinate system in which the origin is at the bottom corner of the triangle, so that h denotes the height of the free surface above this point, the mode-0 and mode-1 solutions are

$$\psi_0 = A\cosh\left(\frac{ky}{\sqrt{2}}\right)\cosh\left(\frac{kz}{\sqrt{2}}\right), \qquad c_0^2 = \frac{g}{\sqrt{2}k}\tanh\left(\frac{kh}{\sqrt{2}}\right),$$

$$\psi_1 = A\sinh\left(\frac{ky}{\sqrt{2}}\right)\sinh\left(\frac{kz}{\sqrt{2}}\right), \qquad c_1^2 = \frac{g}{\sqrt{2}k}\coth\left(\frac{kh}{\sqrt{2}}\right).$$

The symmetric modes $2, 4, 6, \ldots$ are given by

$$\psi = A[\cosh(\alpha y)\cos(\beta z) + \cos(\beta y)\cosh(\alpha z)], \qquad c^2 = -\frac{g\beta}{k^2}\tanh(\beta h), \tag{3.13}$$

where α, β satisfy

$$\alpha^2 - \beta^2 = k^2, \qquad \alpha h\tanh(\alpha h) + \beta h\tan(\beta h) = 0. \tag{3.14}$$

Equations (3.14) have a countably infinite set of solutions $\{(\alpha_n, \beta_n)\}_{n=0}^{\infty}$, each pair (α_n, β_n) of which generates a normal-mode solution through (3.13) followed by (3.3). The correct place of a particular solution in the sequence of normal modes is determined by counting the number of zeros of f in the interval $(0, b)$. The antisymmetric modes $3, 5, 7, \ldots$ are given by

$$\psi = A[\sinh(\alpha y)\sin(\beta z) + \sin(\beta y)\sinh(\alpha z)], \qquad c^2 = \frac{g\beta}{k^2}\coth(\beta h),$$

where α, β satisfy

$$\alpha^2 - \beta^2 = k^2, \qquad \alpha h\coth(\alpha h) - \beta h\cot(\beta h) = 0,$$

and are obtained in the correct sequence using the arguments given above for the symmetric modes. Introducing the non-dimensionalization (3.9), which is here equivalent to setting $g = 1$, $h = 2$ (see figure 4 with $\alpha = \beta = \pi/4$), one may compute the dispersion relations numerically according to the above procedure. This numerical computation is described in detail by Groves (1994, §2.4). The dispersion relations for the first few modes are shown in figure 5.

The symmetric modes for a symmetric triangular channel with sides inclined at an

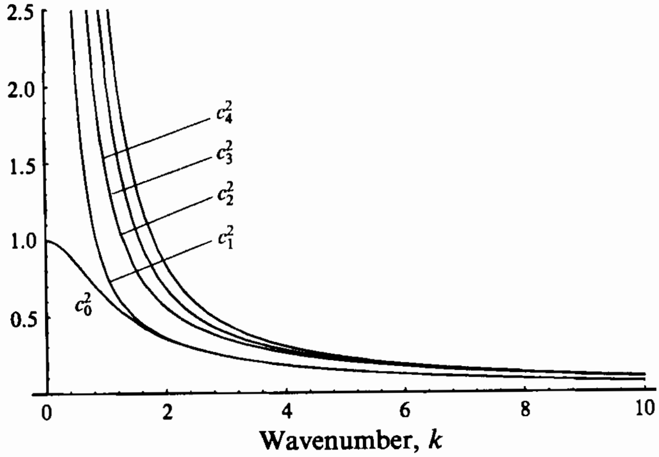

FIGURE 5. Dispersion relations for a symmetric triangular channel of half-angle $\pi/4$.

angle $\pi/3$ to the vertical have been obtained by Packham (1980). The mode-0 and mode-2 solutions had previously been obtained by Macdonald (1894) and are given by

$$\psi = A\left[\cosh k(z-h) + \frac{c^2k^2}{gk}\sinh k(z-h) + 2\cosh\left(\frac{\sqrt{3}ky}{2}\right)\right.$$
$$\left.\times\left\{\cosh k\left(\frac{z}{2}+h\right) - \frac{c^2k^2}{gk}\sinh k\left(\frac{z}{2}+h\right)\right\}\right],$$

where

$$c^2 = \frac{3g}{4k}\coth\left(\frac{3kh}{2}\right)\left\{1 \pm \left[1 - \frac{8}{9}\tanh^2\left(\frac{3kh}{2}\right)\right]^{1/2}\right\}.$$

One chooses the negative sign for the mode-0 solution (because this choice means that f_0 has no zeros) and the positive sign for the mode-2 solution (because this choice means that f_2 has two zeros). The remaining symmetric modes $4, 6, 8, \ldots$ are given by

$$\psi = A\left[\left\{\cosh(\alpha(z-h)) + \frac{c^2k^2}{\alpha g}\sinh(\alpha(z-h))\right\}\cos(\beta y)\right.$$
$$+2\cosh\left(\frac{\sqrt{3}\alpha y}{2}\right)\cos\left(\frac{\sqrt{3}\beta z}{2}\right)\cos\left(\frac{\beta y}{2}\right)$$
$$\times\left\{\cosh\left(\alpha\left(\frac{z}{2}+h\right)\right) - \frac{c^2k^2}{\alpha g}\sinh\left(\alpha\left(\frac{z}{2}+h\right)\right)\right\}$$
$$-2\sinh\left(\frac{\sqrt{3}\alpha y}{2}\right)\sin\left(\frac{\sqrt{3}\beta z}{2}\right)\sin\left(\frac{\beta y}{2}\right)$$
$$\left.\times\left\{\sinh\left(\alpha\left(\frac{z}{2}+h\right)\right) - \frac{c^2k^2}{\alpha g}\cosh\left(\alpha\left(\frac{z}{2}+h\right)\right)\right\}\right],$$

$$c^2 = \frac{g\alpha}{k^2}\left[\frac{(\beta/\alpha)\sqrt{3}(\cosh(3\alpha h) - \cos(\sqrt{3}\beta h))}{(\beta/\alpha)\sqrt{3}\sinh(3\alpha h) - 3\sin(\sqrt{3}\beta h)}\right],$$

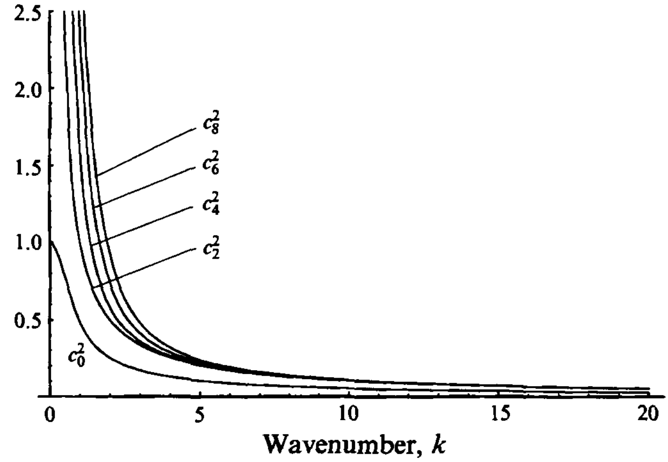

FIGURE 6. Dispersion relations for a symmetric triangular channel of half-angle $\pi/3$.

where α, β satisfy $\alpha^2 - \beta^2 = k^2$ and

$$\left(\frac{\beta}{\alpha}\right)^2 \cosh(3\alpha h)\cos(\sqrt{3}\beta h) - \frac{1}{4}\left(\frac{\beta}{\alpha}\right)\sqrt{3}\left\{1 - \left(\frac{\beta}{\alpha}\right)^2\right\}\sinh(3\alpha h)\sin(\sqrt{3}\beta h)$$
$$-\frac{1}{4}\left[\left\{3+5\left(\frac{\beta}{\alpha}\right)^2\right\} - \left\{3+\left(\frac{\beta}{\alpha}\right)^2\right\}\cos^2(\sqrt{3}\beta h)\right] = 0.$$

The normal modes are obtained from these equations using the argument given above for the triangular channel with half-angle $\pi/4$. Introducing the usual non-dimensionalization (setting $g = 1$, $h = 2$, as shown in figure 4 with $\alpha = \beta = \pi/4$), one may compute the dispersion relations numerically (see Groves 1994, §2.4). The dispersion relations for the first few symmetric modes are shown in figure 6.

The complete set of normal modes for a semicircular channel of radius a have recently been obtained by Evans & Linton (1993). The symmetric modes are found by considering the eigenvalue problem

$$(\boldsymbol{A} - \lambda \boldsymbol{I})p = 0,$$

where $\boldsymbol{A}$ is the infinite matrix with (m, n)th entry

$$A_{mn} = \left[\frac{(m-1)^2 - \frac{1}{4}}{(n-1)^2 - \frac{1}{4}}\right]\left(\frac{2n-1}{ka}\right)\sum_{i=0}^{m-1}\frac{(2-\delta_{i0})I'_{2m-1}(ka)}{I'_{2i}(ka)[(n-\frac{1}{2})^2 - i^2]}.$$

(Here I_j denotes the modified Bessel function of order j.) The symmetric normal modes are given by

$$c_{2m}^2(k) = \frac{\pi^3 ag}{16\lambda_{2m}}, \qquad m = 0, 1, 2, \ldots,$$

$$\psi_{2m}(r, \theta, k) = \sum_{n=0}^{\infty}\frac{(-1)^n p_{2m}^{n+1}}{(n^2 - \frac{1}{4})I'_{2n+1}(ka)}\Bigg(I_{2n}(kr)\cos 2n\theta$$
$$+ I_{2n+2}(kr)\cos(2n+2)\theta - \frac{8kc_{2m}^2}{\pi g}I_{2n+1}(kr)\cos(2n+1)\theta\Bigg), \quad m = 0, 1, 2, \ldots,$$

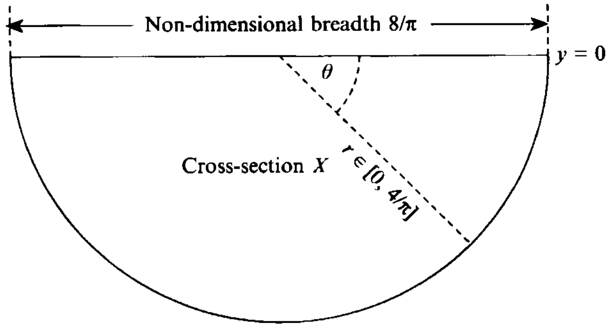

FIGURE 7. A semicircular channel.

in which $\lambda_0, \lambda_2, \ldots$ are the eigenvalues of $\boldsymbol{A}$, p^i_{2m} is the ith component of the eigenvalue corresponding to λ_{2m} and (r, θ) are polar coordinates defined by $z = r\cos\theta$, $y = -r\sin\theta$, where the (y, z) axes have been translated so that the origin is at the centre of the undisturbed free surface.

The asymmetric modes are found by considering the eigenvalue problem

$$(\boldsymbol{B} - \lambda \boldsymbol{I})q = 0,$$

where $\boldsymbol{B}$ is the infinite matrix with (m, n)th entry

$$B_{mn} = \frac{4m(2m-1)}{ka(2n-1)} \sum_{i=1}^{m} \frac{I'_{2m}(ka)}{I'_{2i-1}(ka)[n^2 - (i-\frac{1}{2})^2]}.$$

The asymmetric normal modes are given by

$$c^2_{2m+1} = \frac{\pi^3 ag}{16\lambda_{2m+1}}, \qquad m = 0, 1, 2, \ldots,$$

$$\psi_{2m+1}(r, \theta, k) = \sum_{n=1}^{\infty} \frac{(-1)^n q^n_{2m+1}}{2n(2n-1)I'_{2n+1}(ka)} \bigg(I_{2n-1}(kr)\sin(2n-1)\theta$$
$$+ I_{2n+1}(kr)\sin(2n+1)\theta - \frac{8kc^2_{2m+1}}{\pi g} I_{2n}(kr)\sin 2n\theta \bigg), \qquad m = 0, 1, 2, \ldots$$

in which $\lambda_1, \lambda_3, \ldots$ are the eigenvalues of $\boldsymbol{B}$ and q^i_{2m+1} is the ith component of the eigenvalue corresponding to λ_{2m+1}.

In order to calculate the dispersion relations numerically, one introduces the usual non-dimensionalization (which is equivalent here to setting $g = 1$ and $a = 4/\pi$, as shown in figure 7) and approximates $\boldsymbol{A}$ and $\boldsymbol{B}$ by principal submatrices. Mathematical verification of the validity of this method, together with a discussion of the effect the size of the approximate matrix has on the accuracy of the result, is given by Evans & Linton (1993, §4). The dispersion relations for the first few modes, calculated with the twentieth principal submatrix, are shown in figure 8.

These explicit calculations have recently been supplemented by McIver & McIver (1993, §4), who consider a channel which, in the present notation, is symmetric about $z = b/2$ and present a variational method of calculating upper bounds on the frequencies $kc_0(k)$, $kc_1(k)$, … of the normal modes.

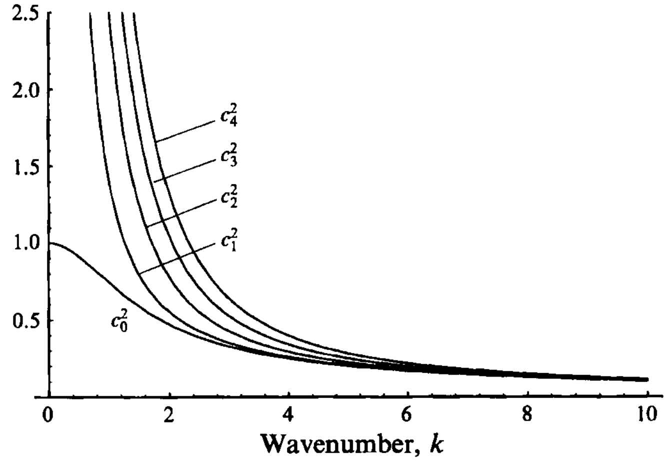

FIGURE 8. Dispersion relations for a semicircular channel.

3.2. *Gravity–capillary waves*

When $\sigma \neq 0$ the mathematical models are described by equations (1.1), (1.2), (1.3), (1.5) together with one of the edge constraints (1.6) or (1.7). Among the classical elementary solutions are sinusoidal solutions of the form

$$\eta(x,z,t) = f(z)\mathrm{e}^{\mathrm{i}k(x-ct)}, \qquad \chi(x,y,z,t) = -\mathrm{i}kc\psi(y,z)\mathrm{e}^{\mathrm{i}k(x-ct)},$$

of (2.10), (2.11), (2.12), (2.13), (2.14), where c, f and ψ depend on the real number k. The functions ψ and f satisfy

$$\psi_{yy} + \psi_{zz} = k^2\psi \quad \text{in} \quad X, \tag{3.15}$$

$$\frac{\partial\psi}{\partial n} = 0 \quad \text{on} \quad \partial X \cap \Gamma, \tag{3.16}$$

$$\left[1 + \frac{\sigma k^2}{g} - \frac{\sigma}{g}\frac{\partial^2}{\partial z^2}\right]\psi_y = f \quad \text{on} \quad \partial X \cap S_0 \tag{3.17}$$

with

$$k^2c^2\psi(0,z) = gf(z) \tag{3.18}$$

and either

$$f(0) = f(b) = 0 \tag{3.19}$$

or

$$f'(0) = f'(b) = 0. \tag{3.20}$$

A solution $\{f, \psi, c^2(k)\}$ of (3.15), (3.16), (3.17), (3.18) together with one of (3.19) or (3.20) is termed a *normal mode* with *dispersion relation* $c^2 = c^2(k)$. To establish the existence and properties of normal modes for a specified channel cross-section one requires a functional-analytic treatment, details of which will be given in a forthcoming article. When surface-tension effects are present, the properties of normal-mode solutions are given by the list in §3.1 with properties (*4*) and (*11*) replaced by

(4) *The set* $\{f_n\}_{n=0}^{\infty}$ *is complete and orthogonal in* $H^1(0,b)$ *(or* $H_0^1(0,b)$ *if the edge constraints (3.19) are imposed), that is*

$$\int_0^b (f_n f_m + f_n' f_m')\,\mathrm{d}z = 0, \qquad n \neq m, \tag{3.21}$$

and any function f *such that* f *and* f' *are square-integrable on* $(0,b)$ *(and also that* $f(0) = f(b) = 0$ *if the edge constraints (3.19) are imposed) may be written as* $f = \sum_{n=0}^{\infty} A_n f_n$*, where the coefficients* A_m *are uniquely determined by* f*;*

(11) c_n^2, $n = 1,2,\ldots$ *has a minimum at some* $k = k_{min} > 0$ *and is monotone decreasing for* $k < k_{min}$ *but monotone increasing for* $k > k_{min}$*;*

(12) c_0^2 *is either monotone increasing for* $k > 0$ *or has a minimum at some* $k = k_{min} > 0$ *and is monotone decreasing for* $k < k_{min}$ *but monotone increasing for* $k > k_{min}$*. The type of behaviour exhibited is determined by the relative sizes of* σ, g *and* d*.*

The only explicit example currently available is for the case of a rectangular channel with the edge constraints (1.7). The normal modes are

$$f_n = A\cos\left(\frac{n\pi z}{b}\right), \qquad n = 0,1,2,\ldots, \tag{3.22}$$

$$\psi_n = A\frac{\cosh\left[\left(k^2 + (n\pi/b)^2\right)^{1/2}(y+h)\right]\cos(n\pi z/b)}{\left(k^2 + (n\pi/b)^2\right)^{1/2}\sinh\left[\left(k^2 + (n\pi/b)^2\right)^{1/2}h\right]}, \qquad n = 0,1,2,\ldots, \tag{3.23}$$

$$c_n^2 = \frac{1}{k^2}\left[g\left(k^2 + (n\pi/b)^2\right)^{1/2} + \sigma\left(k^2 + (n\pi/b)^2\right)^{3/2}\right]\tanh\left[\left(k^2 + (n\pi/b)^2\right)^{1/2}h\right], \qquad n = 0,1,2,\ldots, \tag{3.24}$$

where A is an arbitrary constant and h is the depth of the water in its undisturbed state. In order to plot the dispersion relations one introduces the dimensionless variables

$$(x,y,z) = \frac{1}{d}(x',y',z'), \quad t = t'\left(\frac{g}{d}\right)^{1/2}, \quad \sigma = \sigma'/(gd^2), \quad k = dk', \quad c = c'/(gd)^{1/2}, \tag{3.25}$$

where a prime denotes a dimensional variable. Here d is the mean depth of the undisturbed fluid, so that the non-dimensional area of cross-section A and non-dimensional breadth of the free surface b are equal (figure 2). For a square channel this non-dimensionalization is equivalent to setting $g = h = 1$ in (3.22), (3.23), (3.24). The dispersion relations for a square channel with $\sigma = 0.1$ and $\sigma = 0.4$ are shown in figures 9 and 10 respectively. Notice the difference in the behaviour of the mode-0 dispersion relation for the different values of σ. (In fact c_0^2 is monotone increasing for $k > 0$ if $\sigma/gh^2 \leqslant 1/3$ and has a minimum at some $k = k_{min} > 0$ if $\sigma/gh^2 > 1/3$.)

Notice that (3.15), (3.16), (3.17), (3.18) may be combined into an eigenvalue problem for ψ, namely

$$\psi_{yy} + \psi_{zz} = k^2\psi \qquad \text{in} \qquad X, \tag{3.26}$$

$$\frac{\partial\psi}{\partial n} = 0 \qquad \text{on} \qquad \partial X \cap \Gamma, \tag{3.27}$$

$$\left(1 + \frac{\sigma k^2}{g}\right)\psi_y - \frac{\sigma}{g}\psi_{yzz} = \frac{k^2c^2}{g}\psi \qquad \text{on} \qquad \partial X \cap S_0, \tag{3.28}$$

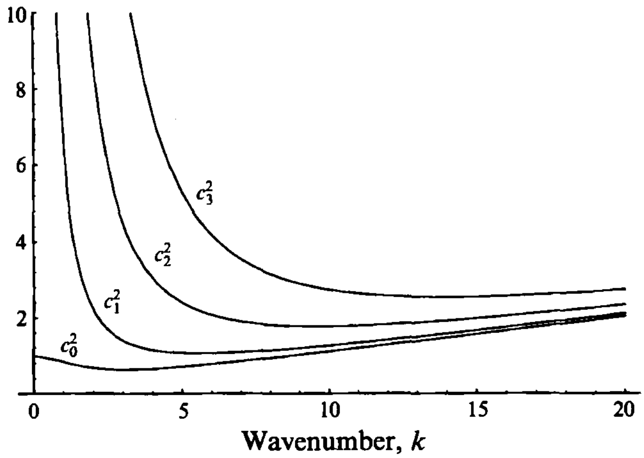

FIGURE 9. Dispersion relations for a square channel with edge constraints (1.7) and $\sigma = 0.1$.

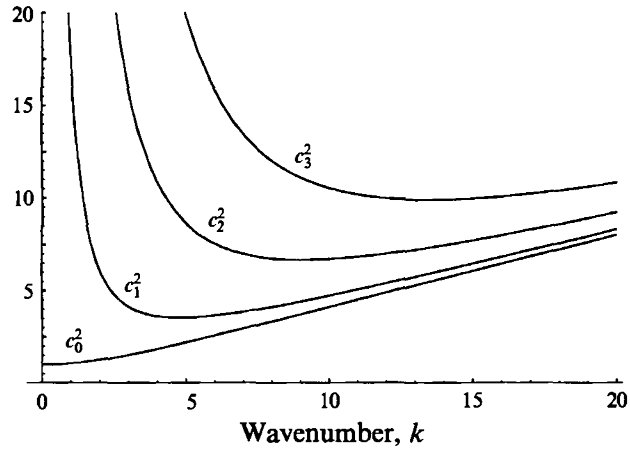

FIGURE 10. Dispersion relations for a square channel with edge constraints (1.7) and $\sigma = 0.4$.

which system is completed by specifying either

$$\psi(0,0) = 0, \qquad \psi(0,b) = 0 \tag{3.29}$$

if one is using the edge constraints (1.6) or

$$\psi_z(0,0) = 0, \qquad \psi_z(0,b) = 0 \tag{3.30}$$

if one is using edge constraints (1.7). This observation indicates a theoretical procedure for calculating the normal modes $(f_n, \psi_n, c_n^2(k))$. One solves one of the above boundary-value problems to determine the doubles $(\psi_n, c_n^2(k))$ and the f_n are then calculated using (3.17) or (3.18).

No explicit solutions for the hydrodynamic problem with the edge constraints (1.6) are currently available. However, the following method may be used to numerically compute the dispersion relations for certain channel cross-sections. (A similar method has been given by Graham-Eagle 1983, where readers may find the details of its functional-analytic foundations. Here only the method itself is described.) Let $\{(\tilde{f}_n, \tilde{\psi}_n, \tilde{c}_n^2(k))\}_{n=0}^{\infty}$ denote the set of normal modes for the hydrodynamic problem with the same geometry but without surface tension. Define

$$N(n) = \int_0^1 (\tilde{f}_n')^2 \,\mathrm{d}z, \qquad n = 0,1,2,\ldots$$

$$\tilde{\mu}_n = \frac{g}{k^2 \tilde{c}_n^2}, \qquad n = 0,1,2,\ldots$$

and without loss of generality assume that the $\tilde{f}_n$ are normalized so that

$$\int_0^1 \tilde{f}_n \,\mathrm{d}z = 1.$$

Subject to the hypothesis that $\{\tilde{f}_n\}_{n=0}^{\infty}$ satisfies not only (3.5) but also (3.21), one has that the function

$$\zeta_S(\omega) = \sum_{m=1}^{\infty} \frac{\left[\left(1+\sigma k^2/g\right) + \sigma N(0)/g - \omega\tilde{\mu}_0\right] (\tilde{f}_{2m}(0))^2}{\left[\left(1+\sigma k^2/g\right) + \sigma N(2m)/g - \omega\tilde{\mu}_{2m}\right] \tilde{f}_0(0)} + \tilde{f}_0(0)$$

is zero at precisely the points $k^2c_0^2/g, k^2c_2^2/g, k^2c_4^2/g, \ldots$ and that the function

$$\zeta_A(\omega) = \sum_{m=1}^{\infty} \frac{\left[\left(1+\sigma k^2/g\right) + \sigma N(1)/g - \omega\tilde{\mu}_1\right] (\tilde{f}_{2m+1}(0))^2}{\left[\left(1+\sigma k^2/g\right) + \sigma N(2m+1)/g - \omega\tilde{\mu}_{2m+1}\right] \tilde{f}_1(0)} + \tilde{f}_1(0)$$

is zero at precisely the points $k^2c_1^2/g, k^2c_3^2/g, k^2c_5^2/g, \ldots$. For any value of k, the values of $c_m^2(k), m = 0,1,\ldots$ may therefore be found by numerically computing the zeros of the functions ζ_S and ζ_A.

In the case of a rectangular channel of depth h and breadth b the functions $\tilde{f}_n$, $N(n)$ and $\tilde{\mu}_n$ are given by

$$\tilde{f}_0 = b^{-1/2}, \qquad \tilde{f}_n = (2/b)^{1/2} \cos\left(\frac{n\pi z}{b}\right), \qquad n = 1,2,3,\ldots, \tag{3.31}$$

$$N(0) = 0, \qquad N(n) = \frac{n^2\pi^2}{b^2}, \qquad n = 1,2,3,\ldots, \tag{3.32}$$

$$\tilde{\mu}_n = \left[k^2 + \left(\frac{n\pi}{b}\right)^2\right]^{-1/2} \coth\left(\left[k^2 + \left(\frac{n\pi}{b}\right)^2\right] h\right), \qquad n = 0,1,2,\ldots. \tag{3.33}$$

The dispersion relations for the first few symmetric normal modes in a square channel are shown in figures 11 and 12. The numerical computations were carried out with non-dimensionalized parameter values $g = h = b = 1$ and $\sigma = 0.1, 0.4$ and approximating the infinite sum by the first forty terms. Comparing figures 11, 12 with figures 9, 10, one finds that the qualitative behaviour of the dispersion relations remains the same, but the phase speeds are higher when the edge constraints (1.6) are imposed.

The present method is applicable to channel geometries where the normal modes in the absence of surface tension are known either explicitly or numerically. A key feature of the method is that it requires $\{\tilde{f}_n\}$ to satisfy both (3.5) and (3.21). In the case of a rectangular channel, by comparing (3.6), (3.7), (3.8) with (3.22), (3.23), (3.24), one finds that the sequences $\{f_n\}$ and $\{\tilde{f}_n\}$ are the same in both cases. Equations (3.5) and (3.21) are therefore both automatically satisfied. One may reasonably conjecture that the two sequences are the same for a wide class of channel geometries. Further justification for this conjecture is given in §5.2.2.

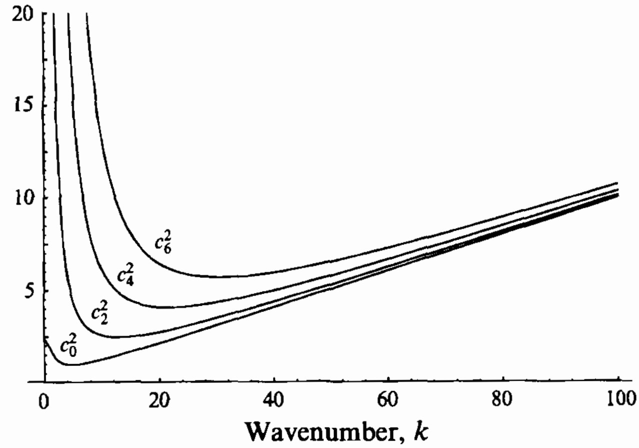

FIGURE 11. Dispersion relations for a square channel with edge constraints (1.6) and $\sigma = 0.1$.

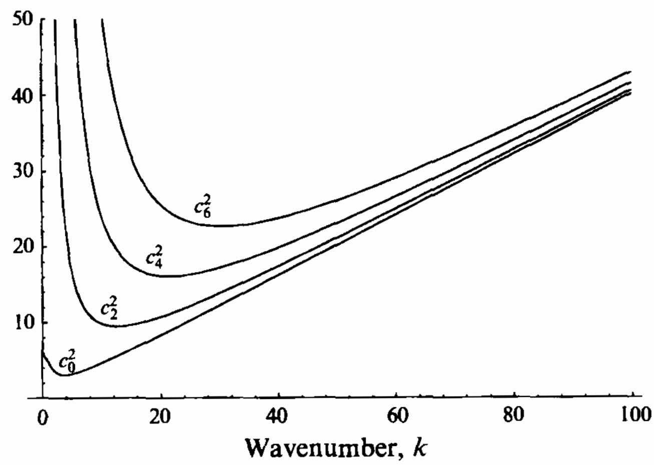

FIGURE 12. Dispersion relations for a square channel with edge constraints (1.6) and $\sigma = 0.4$.

4. The problems in Fourier-transform space

Because the material in this Section is an essential prerequisite for the long-wave theory in §5, only aperiodic wave motions vanishing as $x \to \pm\infty$ are considered here. However, a similar approach for ℓ-periodic waves based upon a Fourier-series decomposition rather than a Fourier transform could presumably be carried out with no extra difficulty.

4.1. *Gravity waves*

Taking the Fourier transform of equations (2.9), one obtains another Hamiltonian evolutionary system, namely

$$\bar{\eta}_t = \frac{\delta \hat{H}_b}{\delta \bar{\Phi}}, \qquad \bar{\Phi}_t = -\frac{\delta \hat{H}_b}{\delta \bar{\eta}}, \tag{4.1}$$

in which

$$\hat{H}_b = \int_0^b \left\{ \tfrac{1}{2} g \bar{\eta}^2 + \tfrac{1}{2} \bar{\Phi} G_2 \bar{\Phi} \right\} \mathrm{d}z$$

and $\bar{\eta}, \bar{\Phi} \in L^2(0,b)$ are the transformed variables

$$\bar{\eta} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, \qquad \bar{\Phi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x.$$

The operator G_2 is also a Dirichlet–Neumann operator, defined as follows. Fix $\bar{\Phi}$, let $\bar{\phi}$ be the unique solution of the elliptic boundary-value problem

$$\bar{\phi}_{yy} + \bar{\phi}_{zz} = k^2\bar{\phi} \quad \text{in} \quad X, \tag{4.2}$$

$$\bar{\phi} = \bar{\Phi} \quad \text{on} \quad \partial X \cap S_0, \tag{4.3}$$

$$\frac{\partial \bar{\phi}}{\partial n} = 0 \quad \text{on} \quad \partial X \cap \Gamma \tag{4.4}$$

and write $G_2\bar{\Phi} = \bar{\phi}_y|_{y=0}$. Here X denotes the interior of the cross-section of D in the (y,z)-plane. (One obtains (4.2), (4.3), (4.4) by taking the Fourier transform of (2.3), (2.4), (2.5).)

Returning to the previous theory concerning normal modes, one finds that each of the pairs

$$\bar{\eta}_n = a_n(k,t)f_n(z,k), \quad \bar{\Phi}_n = b_n(k,t)f_n(z,k), \qquad n = 0,1,2,\ldots \tag{4.5}$$

is a solution of (4.1). Here a_n and b_n are complex-valued functions of k and t related by the formula

$$b_n = -\frac{\mathrm{i}g}{kc_n}a_n.$$

The Hamiltonian system (4.1) thus decomposes into an infinite sequence of independent Hamiltonian systems, the nth of which is

$$\frac{\partial \bar{\eta}_n}{\partial t} = \frac{\delta \hat{H}_n}{\delta \bar{\Phi}_n}, \qquad \frac{\partial \bar{\Phi}_n}{\partial t} = -\frac{\delta \hat{H}_n}{\delta \bar{\eta}_n}, \tag{4.6}$$

with

$$\hat{H}_n = \int_0^b \left\{\tfrac{1}{2}g\bar{\eta}_n^2 + \tfrac{1}{2}\bar{\Phi}_n G_2 \bar{\Phi}_n\right\}\,\mathrm{d}z.$$

It is possible to give an explicit representation of the operator G_2. Because

$$G_2\psi_n(0,z) = \frac{\partial \psi_n}{\partial y}(0,z)$$

for any n, it follows from (3.3), (3.4) that

$$G_2 f_n = \frac{k^2c_n^2(k)}{g}f_n,$$

and therefore

$$G_2\bar{\Phi}_n = \frac{k^2c_n^2(k)}{g}\bar{\Phi}_n.$$

The Hamiltonian $\hat{H}_n$ may consequently be written in the simpler form

$$\hat{H}_n = \int_0^b \left\{\tfrac{1}{2}g\bar{\eta}_n^2 + \frac{k^2c_n^2(k)}{2g}\bar{\Phi}_n^2\right\}\,\mathrm{d}z. \tag{4.7}$$

One more simplification is available. The Hamiltonian system

$$\dot{a}_n = \frac{\partial H_n}{\partial b_n}, \qquad \dot{b}_n = -\frac{\partial H_n}{\partial a_n}, \tag{4.8}$$

in which

$$H_n = \tfrac{1}{2} g a_n^2 + \frac{k^2 c_n^2(k)}{2g} b_n^2, \tag{4.9}$$

is a pair of ordinary differential equations which determines a solution of (4.6) through the relationship (4.5).

4.2. *Gravity–capillary waves*

A similar approach may be adopted when the hydrodynamic problems involving surface-tension effects are under consideration. In this case one takes the Fourier transform of equations (2.15) to obtain another Hamiltonian evolutionary system, namely

$$\bar{\eta}_t = \frac{\delta \hat{H}_c}{\delta \bar{X}}, \qquad \bar{X}_t = -\frac{\delta \hat{H}_c}{\delta \bar{\eta}}, \tag{4.10}$$

where

$$\hat{H}_c = \int_0^b \left\{ \tfrac{1}{2} g \bar{\eta}^2 + \tfrac{1}{2} \bar{X} F_2 \bar{X} \right\} \mathrm{d}z.$$

Here $\bar{\eta}, \bar{X} \in H^1(0,b)$ or $H_0^1(0,b)$ are the transformed variables and F_2 is the operator

$$F_2 = \left[1 + \frac{\sigma k^2}{g} - \frac{\sigma}{g}\frac{\partial^2}{\partial z^2} \right] G_2,$$

in which G_2 is the operator defined in §4.1.

Returning to the previous theory concerning normal modes, one finds that each of the pairs

$$\bar{\eta}_n = a_n(k,t) f_n(z,k), \quad \bar{X}_n = b_n(k,t) f_n(z,k), \qquad n = 0, 1, 2, \ldots, \tag{4.11}$$

where $b_n = -\mathrm{i} g a_n / k c_n$, is a solution of (4.10). The Hamiltonian system (4.10) thus decomposes into an infinite sequence of independent Hamiltonian systems, the nth of which is

$$\frac{\partial \bar{\eta}_n}{\partial t} = \frac{\delta \hat{H}_n}{\delta \bar{X}_n}, \qquad \frac{\partial \bar{X}_n}{\partial t} = -\frac{\delta \hat{H}_n}{\delta \bar{\eta}_n}, \tag{4.12}$$

with

$$\hat{H}_n = \int_0^b \left\{ \tfrac{1}{2} g \bar{\eta}_n^2 + \tfrac{1}{2} \bar{X}_n F_2 \bar{X}_n \right\} \mathrm{d}z.$$

It is possible to give an explicit representation of the operator F_2. Because

$$F_2 \psi_n(0,z) = \left[1 + \frac{\sigma k^2}{g} - \frac{\sigma}{g}\frac{\partial^2}{\partial z^2} \right] \frac{\partial \psi_n}{\partial y}(0,z)$$

for any n, it follows from (3.17), (3.18) that

$$F_2 f_n = \frac{k^2 c_n^2(k)}{g} f_n,$$

and therefore

$$F_2 \bar{X}_n = \frac{k^2 c_n^2(k)}{g} \bar{X}_n.$$

The Hamiltonian $\hat{H}_n$ may consequently be written in the simpler form

$$\hat{H}_n = \int_0^b \left\{ \tfrac{1}{2} g \bar{\eta}_n^2 + \frac{k^2 c_n^2(k)}{2g} \bar{X}_n^2 \right\} \mathrm{d}z. \tag{4.13}$$

Finally notice that the Hamiltonian system (4.8) is a pair of ordinary differential equations which determines a solution of (4.12) through the relationship (4.11).

This section has indicated a theoretical method for solving the initial-value problem for linear water waves in a channel of uniform horizontal cross-section in Fourier-transform space. First one finds the normal modes $\{f_n, \psi_n, c_n^2(k)\}$ for the particular hydrodynamic problem and resolves the initial data $(\bar{\eta}(z,0,k), \bar{\Phi}(z,0,k))$ into its components parallel to each f_n. The evolution of each of these components is then determined by a pair of ordinary differential equations. Although this process is of interest in its own right, the remaining sections of the present paper are more concerned with applications of this theory to other aspects of the hydrodynamic problem, particularly long-wave theory and the transverse wave profiles.

5. Long-wave approximations

5.1. *Derivation of long-wave equations*

Recall that in all three hydrodynamic problems the phase speeds $c(k)$ of all normal modes except the lowest are unbounded in the long-wave limit $k \to 0$. This property implies that the mode-0 component of the solution dominates in the long-wave limit. When carrying out long-wave theory one therefore concentrates on the system (4.8), (4.9) with $n = 0$. The form of the Hamiltonian H_0 suggests a natural approximating scheme for small values of k. One fixes the Hamiltonian structure and the variables a, b and expands the Hamiltonian as a power series in k. Because $c^2(k)$ is an even function of k which is finite at $k = 0$, it has a series expansion in powers of k^2. The Hamiltonian may therefore be expanded in a power series of the form

$$H_0(a_0, b_0, k) = h_0(a_0, b_0) + k^2 h_1(a_0, b_0) + k^4 h_2(a_0, b_0) + \cdots$$

and a sequence of approximations to (4.8) is given by

$$\dot{a}_0 = \frac{\partial}{\partial b_0}\left(\sum_{j=0}^{N} k^{2j} h_j(a_0, b_0)\right), \quad \dot{b}_0 = -\frac{\partial}{\partial a_0}\left(\sum_{j=0}^{N} k^{2j} h_j(a_0, b_0)\right), \qquad N = 0, 1, 2, \ldots$$

This type of *Hamiltonian perturbation theory* has been used on other, more complex Hamiltonian systems to derive long-wave approximations (see Craig & Groves 1994; Groves 1992, 1994).

One may introduce the non-dimensionalization (3.9) to write the expansion of $c^2(k)$ as

$$c_0^2(k) = C_0 + C_1 k^2 + C_2 k^4 + \cdots, \tag{5.1}$$

the dimensional version of which is

$$c_0^2(k) = gd[C_0 + C_1 (kd)^2 + C_2 (kd)^4 + \cdots]. \tag{5.2}$$

The expression for the Hamiltonian is then

$$H_0 = \tfrac{1}{2} g a_0^2 + \tfrac{1}{2} d \left(C_0 k^2 + C_1 d^2 k^4 + C_2 d^4 k^6 + \cdots \right) b_0^2,$$

so that the nth approximate Hamiltonian is

$$H_{A_n} = \tfrac{1}{2} g a_0^2 + \tfrac{1}{2} \left(C_0 d k^2 + C_1 d^3 k^4 + \cdots + C_{n-1} k^{2n} d^{2n-1} \right) b_0^2$$

and the nth approximation to the system (4.8) is

$$\dot{a}_0 = (C_0dk^2 + C_1d^3k^4 + \cdots + C_{n-1}k^{2n}d^{2n-1})b_0, \tag{5.3}$$
$$\dot{b}_0 = -ga_0. \tag{5.4}$$

Equations (5.3), (5.4) determine the longitudinal profile of the wave to any required order of approximation.

The transverse profile is determined by the function $f_0(z,k)$, which may also be expanded in a series of powers of k. The following theory deals with the case when $\sigma = 0$: to obtain the corresponding results for the capillary-wave problems one simply replaces all occurrences of Φ_0 and $\bar{\Phi}_0$ with X_0 and $\bar{X}_0$. Recall that

$$f_0(z,k) = \frac{k^2c_0^2(k)}{g}\psi_0(0,z,k), \tag{5.5}$$

in which the subscript 0 has again been dropped. One may expand ψ as a power series

$$\psi_0(y,z,k) = \xi_0(y,z) + k^2\xi_1(y,z) + k^4\xi_2(y,z) + \cdots. \tag{5.6}$$

Substituting (5.6) and the expansion (5.2) into (5.5), one finds that

$$f_0 = C_0d\xi_0(0,z)k^2 + [C_1d^3\xi_0(0,z) + C_0d\xi_1(0,z)]k^4 + [C_2d^5\xi_0(0,z) + C_0d\xi_2(0,z)]k^6 + \cdots. \tag{5.7}$$

The nth approximation to the transverse wave profile is obtained by retaining terms up to order $2n$ in the above expansion. One may therefore calculate the original Fourier-transformed variables $\bar{\eta}_0, \bar{\Phi}_0$ to $O(k^{2n})$ using the formulae

$$\bar{\eta}_0 = (C_0d\xi_0(0,z)k^2 + \cdots + [C_{n-1}d^{2n-1}\xi_0(0,z) + C_0d\xi_{n-1}(0,z)]k^{2n})a,$$
$$\bar{\Phi}_0 = (C_0d\xi_0(0,z)k^2 + \cdots + [C_{n-1}d^{2n-1}\xi_0(0,z) + C_0d\xi_{n-1}(0,z)]k^{2n})b,$$

where $(a_0, b_0)^T$ is the solution of (5.3), (5.4).

Let us now reformulate these long-wave approximations in terms of partial differential equations. Taking the inverse Fourier transform of (5.3), (5.4), one obtains

$$u_t = -C_0dv_{xx} + C_1d^3v_{xxxx} + \cdots + (-1)^nC_{n-1}d^{2n-1}\frac{\partial^{2n}v}{\partial x^{2n}}, \tag{5.8}$$
$$v_t = -gu, \tag{5.9}$$

in which

$$u(x,t) = \int_{\mathbb{R}} a_0(k,t)e^{ikx}\,dk, \qquad v(x,t) = \int_{\mathbb{R}} b_0(k,t)e^{ikx}\,dk.$$

This system is immediately recognizable as the Hamiltonian evolutionary system

$$u_t = \frac{\delta\hat{H}_n}{\delta v}, \qquad v_t = -\frac{\delta\hat{H}_n}{\delta u}, \tag{5.10}$$

where

$$\hat{H}_n = \int_{\mathbb{R}} \left\{ \tfrac{1}{2}gu^2 + \tfrac{1}{2}dC_0v_x^2 + \tfrac{1}{2}d^3C_1v_{xx}^2 + \cdots + \tfrac{1}{2}d^{2n-1}C_{n-1}\left(\frac{\partial^n v}{\partial x^n}\right)^2 \right\} dx. \tag{5.11}$$

The original variables $\eta_0(x,z,t), \Phi_0(x,z,t)$ are calculated to the present order of approximation using the formulae

$$\eta_0 = \left\{ -C_0 d\xi_0(0,z)\frac{\partial^2}{\partial x^2} + \cdots + (-1)^n [C_{n-1} d^{2n-1}\xi_0(0,z) + C_0 d\xi_{n1}(0,z)]\frac{\partial^{2n}}{\partial x^{2n}} \right\} u, \tag{5.12}$$

$$\Phi_0 = \left\{ -C_0 d\xi_0(0,z)\frac{\partial^2}{\partial x^2} + \cdots + (-1)^n [C_{n-1} d^{2n-1}\xi_0(0,z) + C_0 d\xi_{n1}(0,z)]\frac{\partial^{2n}}{\partial x^{2n}} \right\} v, \tag{5.13}$$

where $(u, v)^T$ is the solution of (5.8), (5.9).

Equations (5.8), (5.9) are the counterparts to the classical long-wave equations for linear surface waves on a two-dimensional expanse of fluid (e.g. see Craig & Groves 1994). However, one should always remember that solutions of (5.8), (5.4) do not predict the wave shape by themselves; these solutions must be related to the physical variables η, Φ by equations (5.12), (5.13). The next section discusses methods of determining the terms $C_0, C_1, \ldots$ and $\xi_0, \xi_1, \ldots$ and calculates the first two pairs (C_0, ξ_0), (C_1, ξ_1) for each of the previously examined channel geometries.

5.2. *Calculation of the terms in the long-wave approximations*

5.2.1. *Gravity waves*

The non-dimensional version of the problem (3.10), (3.11), (3.12) is

$$\psi_{yy} + \psi_{zz} = k^2\psi \quad \text{in} \quad X, \tag{5.14}$$

$$\frac{\partial\psi}{\partial n} = 0 \quad \text{on} \quad \partial X \cap \Gamma, \tag{5.15}$$

$$k^2c^2\psi(0,z) = \psi_y(0,z). \tag{5.16}$$

Integrating (5.14) over X, and using the divergence theorem and (5.15), (5.16), one finds that

$$c^2 = \int_X \psi \,\mathrm{d}y\,\mathrm{d}z \bigg/ \int_0^b \psi(0,z)\,\mathrm{d}z. \tag{5.17}$$

These observations indicate a strategy for systematically determining $\xi_0, \xi_1, \ldots$ and $C_0, C_1, \ldots$. The series expansions (5.6) and (5.1) of ψ_0 and c_0^2 may be substituted into (5.14), (5.15), (5.16), leading to a series of boundary-value problems for $\xi_0, \xi_1, \xi_2, \ldots$. One finds the Nth approximation $C_0 + k^2C_1 + \cdots k^{2(N-1)}C_{N-1}$ to c_0^2 by substituting $\psi_0 = \xi_0 + k^2\xi_1 + \cdots k^{2(N-1)}\xi_{N-1}$ into (5.17) and retaining only terms up to $O(k^{2(N-1)})$ on the right-hand side.

The solutions to the boundary-value problem for ξ_0 are the solutions $\xi_0 =$ constant. One would expect this kind of non-uniqueness because ϕ_m is only defined up to a multiplicative constant. One may remove the non-uniqueness by specifying the value of ξ_0, say $\xi_0 = 1$. Writing $\psi_0 = \xi_0 + O(k^2) = 1 + O(k^2)$ in (5.17), one finds that

$$c_0^2 = \frac{A}{b} + O(k^2) = 1 + O(k^2),$$

so that

$$C_0 = 1.$$

An important consequence of this result is that the lowest-order approximate equations (equations (5.3) and (5.4) with $n = 1$) depend only on the mean depth of the channel. They are always

$$u_t = -dv_{xx}, \qquad v_t = -gu,$$

which equations are analogous to the factored wave equation appearing in classical long-wave theory (e.g. see Lamb 1924, §169). Equation (5.7) indicates that

$$f_0 = dk^2 + O(k^4),$$

so that η and Φ are obtained from u and v by differentiating twice with respect to x and multiplying by $-d$. To this order of approximation the wave therefore has no transverse variation in its shape.

The problem for ξ_1 is

$$\xi_{1yy} + \xi_{1zz} = 1 \qquad \text{in} \qquad X, \tag{5.18}$$

$$\frac{\partial \xi_1}{\partial n} = 0 \qquad \text{on} \qquad \partial X \cap \Gamma, \tag{5.19}$$

$$\xi_{1y} = 1 \qquad \text{on} \qquad y = 0. \tag{5.20}$$

The solution is determined up to an arbitrary function of x and t but may be made unique by specifying the value of ξ_1 at a point on $\partial X \cap \Gamma$, say

$$\xi_1(0, b/2) = 0. \tag{5.21}$$

Writing $\psi_0 = 1 + k^2\xi_1 + O(k^4)$ in (5.17) and noting that $A = b$, one finds that

$$C_1 = \frac{1}{b}\left[\int_X \xi_1 \, \mathrm{d}y \, \mathrm{d}z - \int_0^b \xi_1(0, z) \, \mathrm{d}z\right]. \tag{5.22}$$

The number C_1 has previously been termed the *channel number* in Groves (1992 §3.5). It depends only on the geometry of the channel, and is unique, even though ξ_1, without the imposition of (5.21), is not. The second-order long-wave approximation is given by equations (5.3), (5.4) with $n = 2$ together with the expansion

$$f_0 = (dk^2 + C_1 d^3 k^4) + d\xi_1(0, z)k^4 + O(k^6).$$

The transverse profile of a wave to the present order of approximation is therefore described by $\xi_0(0, z)$ plus a constant.

For a rectangular channel of non-dimensional height 1 and non-dimensional breadth b (figure 2) one finds that

$$\xi_1 = \tfrac{1}{2}y^2 + y, \qquad C_1 = -\tfrac{1}{3}.$$

Observe that $\xi_1(0, z) = 0$, so that there is no transverse variation in the wave profile. The corresponding results for a triangular channel of non-dimensional height 2 (figure 4) are

$$\xi_1 = \tfrac{1}{4}(y^2 + z^2) - 1, \qquad C_1 = -\tfrac{1}{6}(\tan^2\alpha - \tan\alpha\tan\beta + \tan^2\beta) - \tfrac{1}{2}.$$

Finally, Peters (1966) has computed these functions for a semicircular channel of non-dimensional radius $a = 4/\pi$ (figure 7). The results are

$$\xi_1(y, z) = \tfrac{1}{4}(y^2 + z^2) + y$$
$$\qquad - \frac{a}{\pi}\mathrm{Re}\left[\frac{(w + a)^2}{(w + a)}\log(a + w) - \frac{(w - a)^2}{aw}\log(a - w) - 2 - 4\log a\right],$$

$$C_1 = \frac{3\pi^2 - 48\log 2 - 74}{3\pi^2}.$$

In cases where (5.18), (5.19), (5.20), (5.21) cannot be solved explicitly, one may make use of the variational principle

$$C_1 = \frac{2}{b} \min_{\zeta} \left[\int_X \left\{ \tfrac{1}{2}|\nabla\zeta|^2 + \zeta \right\} \mathrm{d}y \, \mathrm{d}z - \int_0^b \zeta(0,z) \, \mathrm{d}z \right] \tag{5.23}$$

where the competitors for the minimum belong to the space $H^1(X)$ of functions ζ such that ζ and ζ' are square integrable on X and satisfy $\zeta(0, b/2) = 0$ (see Groves 1994, §2.8).

The variational principle (5.23) is useful for the purpose of estimating C_1. Any function $\zeta \in H^1(X)$ that satisfies $\zeta(0, b/2) = 0$ may be substituted into (5.23) to give an upper bound on C_1. An important consequence of this fact is that C_1 is negative, which result follows by substituting $\zeta = 0$ into (5.23). The variational principle may also be useful for numerical computations of the channel number, particularly by finite-element methods (see Groves 1994, §2.8 for a discussion of this point).

5.2.2. *Gravity–capillary waves with edge constraints (1.7)*

Since it has the greater number of similarities with the case when surface tension is absent, let us tackle the problem with edge constraints (1.7) first. The non-dimensional versions of equations (3.26), (3.27), (3.28), (3.30) are

$$\psi_{yy} + \psi_{zz} = k^2 \psi, \tag{5.24}$$

$$\frac{\partial \psi}{\partial n} = 0 \quad \text{on} \quad \partial X \cap \Gamma, \tag{5.25}$$

$$(1 + \sigma k^2)\psi_y - \sigma \psi_{yzz} = k^2 c^2 \psi \quad \text{on} \quad \partial X \cap S_0, \tag{5.26}$$

$$\psi_z(0,0) = 0, \tag{5.27}$$

$$\psi_z(0,b) = 0. \tag{5.28}$$

Observe first that a little more information about the edge constraints may be gleaned from (5.24), (5.25), (5.26), (5.27), (5.28). Let s denote arclength around $\partial X \cap \Gamma$. Because the normal derivative of ψ is always zero here, it follows that

$$\frac{\partial}{\partial s}\left(\frac{\partial \psi}{\partial n}\right) = 0 \quad \text{on} \quad \partial X \cap \Gamma.$$

Using this piece of information, together with (5.27), (5.28), one finds that

$$\psi_{yz}(0,0) = \psi_{yz}(0,b) = 0. \tag{5.29}$$

Let us now follow the theory in §5.2.1. Integrating (5.24) over X and using the divergence theorem, (5.25), (5.26) and (5.29), one finds that

$$c^2 = (1 + \sigma k^2) \int_X \psi \, \mathrm{d}y \, \mathrm{d}z \bigg/ \int_0^b \psi(0,z) \, \mathrm{d}z \tag{5.30}$$

One now substitutes the series expansions

$$\psi_0 = \xi_0 + k^2 \xi_1 + k^4 \xi_2 + \cdots,$$
$$c_0^2 = C_0 + k^2 C_1 + k^4 C_2 + \cdots$$

for the non-dimensionalized variables ψ and c_0^2 into (5.24), (5.25), (5.26), (5.29), leading to a series of well-posed boundary-value problems for $\xi_0, \xi_1, \xi_2, \ldots$. One finds the Nth approximation $C_0 + k^2 C_1 + \cdots k^{2(N-1)} C_{N-1}$ to c_0^2 by substituting $\psi_0 =$

$\xi_0 + k^2\xi_1 + \cdots k^{2(N-1)}\xi_{N-1}$ into (5.30) and retaining only terms up to $O(k^{2(N-1)})$ on the right-hand side.

The solutions for the boundary-value problem for ξ_0 are the solutions ξ_0 = constant. As before, one removes the uniqueness by choosing $\xi_0 = 1$. Writing $\psi_0 = \xi_0 + O(k^2) = 1 + O(k^2)$ in (5.30), one finds that

$$c_0^2 = \frac{A}{b} + O(k^2) = 1 + O(k^2),$$

so that, as before,

$$C_0 = 1.$$

The problem for ξ_1 is

$$\xi_{1yy} + \xi_{1zz} = 1 \quad \text{in} \quad X, \tag{5.31}$$

$$\frac{\partial \xi_1}{\partial n} = 0 \quad \text{on} \quad \partial X \cap \Gamma, \tag{5.32}$$

$$\xi_{1y} - \sigma\xi_{1yzz} = 1 \quad \text{on} \quad y = 0, \tag{5.33}$$

$$\xi_{1yz}(0,0) = 0, \tag{5.34}$$

$$\xi_{1yz}(0,b) = 0. \tag{5.35}$$

An elementary calculation shows that equations (5.33), (5.34), (5.35) are satisfied simultaneously if and only if

$$\xi_{1y} = 1 \quad \text{on} \quad y = 0.$$

The term ξ_1 is therefore the same as in the problem with no surface tension. It is determined up to an arbitrary function of x and t, but as before may be made unique by specifying $\xi_1(0, b/2) = 0$. Writing $\psi_0 = 1 + k^2\xi_1 + O(k^4)$ in (5.30), one finds that

$$c_0^2 = 1 + \frac{k^2}{b}\left[\sigma + \int_X \xi_1 \,\mathrm{d}y\,\mathrm{d}z - \int_0^b \xi_1(0,z)\,\mathrm{d}z\right] + O(k^4),$$

which result means that

$$C_1 = \sigma + \frac{1}{b}\left[\iint_X \xi_1 \,\mathrm{d}y\,\mathrm{d}z - \int_0^b \xi_1(0,z)\,\mathrm{d}z\right].$$

The coefficient C_1 is therefore differs from the channel number which appears in the problem without surface tension only by the addition of σ.

Observe that the terms ξ_0, ξ_1 in the expansion of ψ_0 are the same as those in the corresponding expansion for the hydrodynamic problem in the absence of surface tension (§5.2.1). It is easy to see that the remaining terms $\xi_2, \xi_3, \ldots$ are also the same in both cases. For a discussion of the significance of this point, see the remarks just before the start of §4.

5.2.3. *The hydrodynamic problem with edge constraints (1.6)*

A different approach is needed to compute the coefficients $C_0, C_1, C_2, \ldots$ and functions $\xi_0, \xi_1, \xi_2, \ldots$ for this version of the hydrodynamic problem. Recall from §3.2 that $\omega = k^2c_0^2/g$ is the first root of

$$\sum_{m=1}^{\infty} \frac{[(1+\sigma k^2/g) + \sigma N(0)/g - \omega\tilde{\mu}_0]\,(\tilde{f}_{2m}(0))^2}{[(1+\sigma k^2/g) + \sigma N(2m)/g - \omega\tilde{\mu}_{2m}]\,\tilde{f}_0(0)} + \tilde{f}_0(0) = 0. \tag{5.36}$$

One may substitute the expansion

$$w = \frac{k^2}{g}c_0^2(k) = dk^2[C_0 + C_1(kd)^2 + C_2(kd)^4 + \cdots]$$

of w in (5.36) and equate equal powers of k^2 to successively determine $C_0, C_1, C_2, \ldots$

The values of the functions $f_m, \tilde{\mu}_m, N(m)$ for a rectangular channel are given by equations (3.31), (3.32), (3.33). One finds after a long but straightforward calculation that

$$C_0 = \left[1 - \frac{\tanh(gb^2/4\sigma)^{1/2}}{(gb^2/4\sigma)^{1/2}}\right]^{-1},$$

$$C_1 = \frac{1}{6gh} - \frac{\sigma}{2g^2h^3} + \frac{1}{gh^3C_0^2}(C_0^2 - 1)^2 \left\{ \frac{b\sigma^{1/2}}{8g^{1/2}} \coth\left((gb^2/4\sigma)^{1/2}\right) \right.$$
$$\left. - \frac{b^2}{16}\operatorname{cosec}^2\left((gb^2/4\sigma)^{1/2}\right) + hC_0^2 \sum_{m=1}^{\infty} \frac{g^2 b \coth(2m\pi/b)}{2\sigma^2 m\pi(g/\sigma + 4m^2\pi^2/b^2)^2} \right\}$$

(see also Benjamin & Graham-Eagle 1985, p. 105).

Having computed $C_0, C_1, C_2, \ldots$, one may substitute the series

$$c_0^2(k) = C_0 + C_1k^2 + C_2k^4 + \cdots,$$
$$\psi_0(y,z,k) = \xi_0(y,z) + k^2\xi_1(y,z) + k^4\xi_2(y,z) + \cdots$$

into the non-dimensionalized versions of equations (3.26), (3.27), (3.28), (3.29), leading to a sequence of boundary-value problems for $\xi_0, \xi_1, \xi_2, \ldots$. These boundary-value problems are not trivial to solve, even for a rectangular channel, and may be tractable only by numerical solution.

6. Bifurcation theory for travelling waves

Let us conclude by briefly considering the hydrodynamic problems for travelling waves, that is solutions of the form $\eta(x,z,t) = \eta(x-ct,z)$, $\phi(x,y,z,t) = \phi(x-ct,y,z)$. The system of equations obtained by substituting this ansatz for η and ϕ into the equations of motion may be regarded as a dynamical system in which x is the time-like variable and c is a parameter. (A precise discussion of this point in the context of the classical water-wave problem is given by Groves & Toland 1995, where it is shown that the dynamical system in question is Hamiltonian. A forthcoming paper will provide the corresponding generalization to the channel setting.)

One proceeds by analysing the spectrum of this dynamical system. The number and magnitude of the purely imaginary non-zero eigenvalues $\pm ik$, which correspond to bounded $2\pi/k$-periodic travelling waves, depend on the value of c, and this information is of course contained in the diagrams of the dispersion relations (figures 3, 5, 6, 8 when $\sigma = 0$, figures 9, 10, 11, 12 when $\sigma > 0$). The curves in these diagrams indicate the values of k corresponding to purely imaginary eigenvalues $\pm ik$. Moreover, the curves have the same qualitative properties regardless of the shape of the channel. These generic features may be exploited in order to deduce qualitative information concerning periodic travelling waves.

One studies the diagrams of the dispersion relations by regarding c as a bifurcation parameter and considering the development of periodic waves of different wavenumbers as c is increased. Let us first consider the capillary–gravity wave problems, as represented by figure 9. There is a minimum value $c_0^\star = \min_{k>0} c_0$ of c, below which

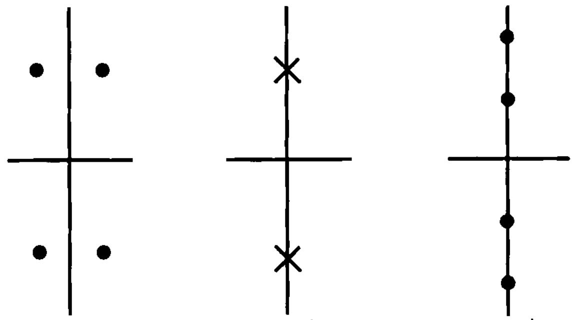

FIGURE 13. Bifurcation of two periodic waves at $c = c_0^*$.

there are no non-trivial periodic waves. When c is increased through c_0^*, two periodic solutions appear: a periodic wave whose wavenumber decreases as c is increased, so that it becomes a gravity wave; and a periodic wave whose wavenumber increases as c is increased, so that it becomes a capillary wave. The value c_0^* is a bifurcation point at which these two periodic solutions appear. Here two pairs of complex eigenvalues collide and separate on the imaginary axis (see figure 13). Further bifurcations of the same type occur as c is increased through the bifurcation points $c_n^* = \min_{k>0} c_n$.

Observe that the situation is slightly different when the physical parameters g, b and σ are configured such that c_0^* occurs at $k = 0$ (figure 10). In this case only one wave is generated as c passes through the bifurcation point c_0^*, namely the wave which is ultimately dominated by capillary effects. Here a pair of real eigenvalues collides at the origin and separates on the imaginary axis. Finally notice that when $\sigma = 0$ there is an infinite number of periodic travelling waves for each value of c (figures 3, 5, 6, 8). A bifurcation analysis based upon the above ideas is therefore not possible.

The above study of the qualitative properties of the diagrams depicting the dispersion relations sets in place the theoretical framework for an analysis of the nonlinear generalizations of the channel problems with a view to constructing a bifurcation theory for periodic travelling-wave solutions. Such an analysis has been undertaken for the classical two-dimensional water-wave problem by Jones & Toland (1986) and Jones (1989). A forthcoming paper will extend the ideas of these authors to the nonlinear channel problems using the above framework.

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REFERENCES

BENJAMIN, T. B. 1980 Theoretical problems posed by gravity–capillary waves with edge constraints. In *Trends in Applications of Pure Mathematics to Mechanics III* (ed. R. J. Knops), pp. 40–58. Pitman.

BENJAMIN, T. B. 1984 Impulse, flow-force and variational principles. *IMA J. Appl. Maths* **32**, 3–68.

BENJAMIN, T. B. & GRAHAM-EAGLE, J. 1985 Long gravity–capillary waves with edge constraints. *IMA J. Appl. Maths* **35**, 91–114.

BENJAMIN, T. B. & OLVER, P. J. 1982 Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.* **125**, 137–185.

BENJAMIN, T. B. & SCOTT, J. C. 1979 Gravity-capillary waves with edge constraints. *J. Fluid Mech.* **92**, 241–267.

CRAIG, W. 1991 Water waves, Hamiltonian systems and Cauchy integrals. In *Microlocal Analysis and Nonlinear Waves* (ed. M. Beals, R. B. Melrose & J. Rauch), pp. 37–45. Springer-Verlag.

Craig, W. & Groves, M. D. 1994 Hamiltonian long-wave approximations to the water-wave problem. *Wave Motion* **19**, 367–389.

Dussan V., E. B. 1979 On the speading of liquids on solid surfaces: static and dynamic contact lines. *Ann. Rev. Fluid Mech.* **11**, 371–400.

Evans, D. V. & Linton, C. M. 1993 Sloshing frequencies. *Q. J. Mech. Appl. Maths* **46**, 71–87.

Graham-Eagle, J. 1983 A new method for calculating eigenvalues with applications to gravity–capillary waves with edge constraints. *Math. Proc. Camb. Phil. Soc.* **94**, 553–564.

Groves, M. D. 1992 Hamiltonian long-wave approximations for water waves in a uniform channel. In *Nonlinear Dispersive Wave Systems* (ed. L. Debnath), pp. 99–125. World Scientific.

Groves, M. D. 1994 Hamiltonian long-wave theory for water waves in a channel. *Q. J. Mech. Appl. Maths* **47**, 367–404.

Groves, M. D. & Toland, J. F. 1995 On variational formulations for steady water waves. Preprint available by anonymous ftp from ftp.maths.bath.ac.uk: /pub/preprints.

Hocking, L. M. 1987 The damping of capillary–gravity waves at a rigid boundary. *J. Fluid Mech.* **179**, 253–266.

Jones, M. 1989 Small amplitude gravity–capillary waves in a channel of finite depth. *Glasgow Math. J.* **31**, 141–160.

Jones, M. & Toland, J. F. 1986 Symmetry and the bifurcation of capillary–gravity waves. *Arch. Rat. Mech. Anal.* **96**, 29–53.

Lamb, H. 1924 *Hydrodynamics,* 5th edn. Cambridge University Press.

Macdonald, H. M. 1894 Waves in canals. *Proc. Lond. Math. Soc.* **25**, 101–111.

McIver, P. & McIver, M. 1993 Sloshing frequencies of longitudinal modes for a liquid contained in a trough. *J. Fluid Mech.* **252**, 525–541.

Miles, J. W. 1977 On Hamilton's principle for surface waves. *J. Fluid Mech.* **83**, 153–158.

Miles, J. W. 1981 Hamiltonian formulations for surface waves. *Appl. Sci. Res.* **37**, 103–110.

Packham, B. A. 1980 Small-amplitude waves in a straight channel of uniform triangular cross-section. *Q. J. Mech. Appl. Maths* **33**, 179–187.

Peters, A. S. 1966 Rotational and irrotational solitary waves in a channel with arbitrary cross-section. *Commun. Pure Appl. Maths* **19**, 445–471.

Scott, J. C. & Benjamin, T. B. 1978 Waves in narrow channels: faster capillary waves. *Nature* **276**, 803–805.

Shen, M. C. 1983 Nonlinear capillary waves under gravity with edge constraints in a channel. *Phys. Fluids* **26**, 1417–1421.

Weidman, P. D. & Norris, J. A. 1987 Capillary gravity waves with fixed contact lines: an approximate analysis. *Physicochem. Hydrodyn.* **9**, 393–402.

Whitham, G. B. 1974 *Linear and Nonlinear Waves*. Wiley-Interscience.

Zakharov, V. E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Zh. Prikl. Mekh. Tekh. Fiz.* **9**, 86–94. (English translation *J. Appl. Mech. Tech. Phys.* **2**, 190–194.)